Domino Tilings of a Rectangle Grid with a Missing Cell

Askar Dzhumadil'dayev, Kazakh-British Technical University
Radmir Sultamuratov, Institute of Modelling and Mathematics

June 2016

Abstract

In this article we work out the algorithm of calculating domino tilings of $m \times n$ rectangle with some removed squares. Domino tilings in rectangle $m \times n$ are also known as perfect matchings of graph $P_m \times P_n$ or as dimer problem. Research on dimers in statistical mechanics had a major breakthrough in 1961, when Kasteleyn (and, independently, Temperley&Fisher) discovered a Pfaffian method to count the matchings of subgraph of the infinite square lattice and introduced remarkable formula $\prod_{j=1}^{\lceil \frac{m}{2} \rceil} \prod_{k=1}^{\lceil \frac{n}{2} \rceil} \left(4cos^2 \frac{\pi j}{m+1} + 4cos^2 \frac{\pi k}{n+1}\right)$, which calculates the number of all possible domino tilings of $m \times n$ rectangle, for case when mn is even. Since there is no perfect matching for case, when mn is odd, the first goal for us was to study the number of all possible maximum matchings for this case, whereas it is no different from domino tilings of the rectangle $m \times n$ with one removed square. Our calculating method is similar to the method of David Klarner and Jordan Polack, which is calculating domino tilings of rectangles with fixed width and differs completely from the known Pfaffian method.

Keywords: domino tilings, dimers, maximum matchings, fixed width, adjacency matrices, $P_m \times P_n$

1 Introduction

Previously, David Klarner and Jordan Polack [1] have constructed bijection between set of domino tilings of rectangles with fixed width and class of paths in certain graph. Also they used adjacency matrices to calculate the cardinality of this class.

Consider a $m \times n$ tiling, and consider the vertical line segments of length m a unit apart which cut across the rectangle into columns. Every such cross section is encoded as a vector $u = (u_1, \dots, u_m) \in \mathbb{Z}_2^m$ as follows: $u_i = 1$ if the line cuts through a horizontal domino in i^{th} row, otherwise $u_i = 0$. So every domino tiling of rectangle $m \times n$ corresponds to a sequence of $\mathbb{Z}_2^m$ vectors of length $n + 1$. Since no domino lies on the boundaries of the rectangle, the first and last cross sections are zero vectors. Let $\mathbb{Z}_2^m$ be the vertex set of G_m, and let

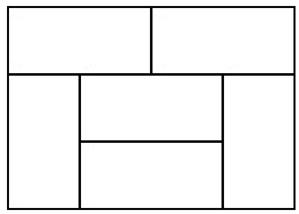


Figure 1: Cross sections (from left to right):(000),(100),(011),(100),(000)

$u, v \in \mathbb{Z}_2^m$. Then (u, v) forms a directed edge in G_m just when two successive cuts in a strip m units wide tiled with dominoes gives rise to u and v going from left to right. (Note that u and v may occur in a strip tiled with dominoes, but not necessarily in a rectangle tiled with dominoes.) It follows from these definitions that there is a one-to-one correspondence between paths of length n which begin and end at zero vectors in G_m and the $m \times n$ tilings.

In our work we extend this approach for rectangles with removed squares.

2 Column Tilings

Definition 2.1. Let E_m be the set of edges of graph G_m as it was defined in the previous section.

Every edge in E_m is a pair of left and right cross sections of column tiled by dominoes. Left cross section indicates horizontal dominoes crossing left border of this column and the same on the right side. All the rest is covered by vertical dominoes.

Definition 2.2. Let columns with width m covered by dominoes as described above be called m-tilings.

In this section, we correspond every edge $(u, v) \in E_m$ to an m-tilings with left cross section u and right cross section v. So set E_m can be also considered as a set of m-tilings.

Definition 2.3. Let $E_{\overline{m}}$ be a subset of E_m containing only one edge: loop that connects zero vector of $\mathbb{Z}_2^m$ to itself.

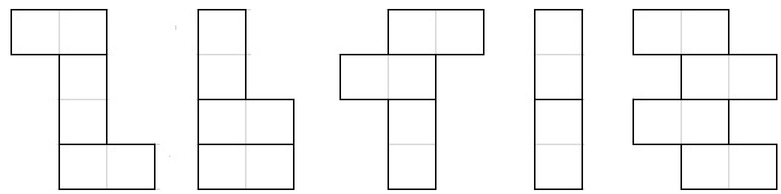


Figure 2: 4-tilings (from left to right):
(1000,0001), (0000,0011), (0100,1000), (0000,0000), (1010,0101)

In other words, $\overline{m}$-tiling implies a column of width m with $\vec{0}$ cross sections from the both sides, so no domino can intersect this column. This part we refer to vertical line of removed squares.

Definition 2.4. Let $\mu_1|\mu_2|\cdots|\mu_k$-tiling be a vertical join (one under another) of μ_i-tilings, where every μ_i-tiling is m_i-tiling or $\overline{m_i}$-tiling. Let $\mu_1|\mu_2|\cdots|\mu_k$ be called a class of column tilings.

In this way we explain columns with some removed squares: $\overline{m_i}$ is a part of column which is removed (not tiled by dominoes), whereas m_i is a part which is tiled by dominoes in free way.

For example, $\overline{1}|3|\overline{2}|1$ represents set of domino tilings of 7 width column with $1^{st}, 4^{th}$ and 5^{th} squares removed.

Definition 2.5. Let $E_{\mu_1|\mu_2|\cdots|\mu_k}$ be a set of all edges corresponding to $\mu_1|\mu_2|\cdots|\mu_k$-tilings.

In this section, edges of G_m have been considered as m-tilings. As well some classes of m-tilings have been highlighted.

As it was said we investigate paths in graph G_m to calculate number of domino tilings. In the case when one or some squares are removed from rectangle, paths are still paths, but with certain limitations (in certain intervals), because no domino can cover removed squares. Later, we will calculate paths in the graph G_m with periodically changing set of edges.

3 Adjacency Matrices

Definition 3.1. Let A_μ be an adjacency matrix of graph $G(\mathbb{Z}_2^m, E_\mu)$, where μ is a class of m-tilings.

Theorem 3.1. *If μ and ν be classes of column tilings, then*

$$A_{\mu|\nu} = A_\mu \otimes A_\nu$$

Proof. Both sides are binary matrices of order $2^{|\mu|+|\nu|}$. Let (u, v) entry of $A_{\mu|\nu}$ on LHS be the product of (u_1, v_1) entry of A_μ and (u_2, v_2) entry of A_ν on RHS.

By the definition 3.1 (u, v) entry of $A_{\mu|\nu}$ is "1" if $(u, v) \in E_{\mu|\nu}$, otherwise (u, v) entry is "0".

Let's say that (u, v) entry is "1". Then according to the definition 2.5 (u, v) is $\mu|\nu$-tiling. That is true if only (u_1, v_1) is μ-tiling and (u_2, v_2) is ν-tiling. Which means that (u_1, v_1) and (u_2, v_2) entries are both "1" and their product is "1" too.

If (u, v) entry is "0". Then (u, v) is not $\mu|\nu$-tiling and both of (u_1, v_1) and (u_2, v_2) cannot be μ-tiling and ν-tiling, respectively. Which means that at least one of (u_1, v_1) and (u_2, v_2) entries is "0". So their product is "0" too.

So LHS and RHS are equal matrices. ■

Corollary 3.1.1. $A_{\mu_1|\mu_2|\cdots|\mu_k} = A_{\mu_1} \otimes A_{\mu_2} \otimes \cdots \otimes A_{\mu_k}$

Corollary 3.1.2. $A_0 = [1]$

Proof. A_0 is supposed to be $2^0 \times 2^0$ matrix. Moreover, $A_\mu = A_{\mu|0} = A_\mu \otimes A_0$. ■

Theorem 3.2. $A_m = A_{m_1|m_2} + A_{m_1-1|\bar{2}|m_2-1}$, *where* $m = m_1 + m_2$.

Proof. It is enough to show that $E_m = E_{m_1|m_2} \bigcup E_{m_1-1|\bar{2}|m_2-1}$.

Set of m-tilings can be divided into two subsets: 1) m-tilings with a vertical domino covering both of m_1^{th} and $(m_1+1)^{th}$ squares of column; 2) and without vertical domino on the same place. First subset is set of $m_1|m_2$-tilings and second is $m_1-1|\bar{2}|m_2-1$-tilings. ■

Corollary 3.2.1. $A_{m+2} = A_1 \otimes A_{m+1} + A_{\bar{2}} \otimes A_m$.

Proof. From theorem 3.2 for $m_1 = 1$ it follows that $A_{m+2} = A_{1|m+1} + A_{\bar{2}|m}$. And from corollary 3.1.1 it follows that $A_{1|m+1} = A_1 \otimes A_{m+1}$ and $A_{\bar{2}|m} = A_{\bar{2}} \otimes A_m$. ■

Remark. Let vertices in graph $G(\mathbb{Z}_2^m, E_\mu)$ be ordered and numbered in lexicographical order starting with $\vec{0}$ and ending with $\vec{1}$ for any μ.

Proposition 3.3. $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_{\bar{1}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, A_{\bar{2}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof. Since $E_1 = \{(\{1\},\{0\}),(\{0\},\{1\})\}$, A_1 is 2×2 matrix with two "1".

For any $m \in \mathbb{N}$ matrix $A_{\overline{m}}$ is $2^m \times 2^m$ matrix with "1" in $(1,1)$ entry and "0" in all other entries, because $E_{\overline{m}} = \{(\vec{0},\vec{0})\}$.

According to corollaries 3.2.1 and 3.1.2

$$A_2 = A_1 \otimes A_1 + A_{\overline{2}} \otimes A_0 =$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \otimes [1] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

■

Remark. Using recurrence relation (corollary 3.2.1) and initial data (proposition 3.3) we can calculate matrix A_m for any $m \in \mathbb{N}$:

$$A_{m+2} = \left[\begin{array}{c|c} \begin{array}{c|c} A_m & 0_{2^m,2^m} \\ \hline 0_{2^m,2^m} & 0_{2^m,2^m} \end{array} & A_{m+1} \\ \hline A_{m+1} & 0_{2^{m+1},2^{m+1}} \end{array}\right]$$

Then using corollary 3.1.1 we can calculate matrix A_μ for any μ.

For example,

$$A_{1|2|\overline{1}} = A_1 \otimes A_2 \otimes A_{\overline{1}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

4 Domino Tilings and Paths

Let's consider a column C with one or some removed squares in strip m units wide tiled with dominoes. Let $n_0, r_1, n_1, \dots r_k, n_k$ be an alternating sequence of removed and non-removed squares in this column C, where r_i is length of i_{th} continuous sequence of removed squares and n_i of non-removed. $n_0 = 0$ if first square turned out to be removed and $n_k = 0$ if last square turned out to be removed. All other n_i and r_i are positive. Clearly, all possible domino tilings of this column are $n_0|\overline{r_1}|n_1| \dots |\overline{r_k}|n_k$-tilings.

Further, for convenience we will denote all possible domino tilings of column C as C-tilings. Respectively, E_C is a set of edges corresponding to C-tilings and A_C is adjacency matrix of graph $G(\mathbb{Z}_2^m, E_C)$. The point is that we know how to calculate matrix A_C for any column C.

Theorem 4.1. *Let R be $m \times n$ rectangle with one or some removed squares. Then $(1,1)$ entry of the matrix $A_{C_1} A_{C_2} \cdots A_{C_n}$ is equal to the number of all possible domino tilings of the rectangle R, where C_i is i^{th} column of the rectangle R.*

Proof. As it was said before, there is one-to-one correspondence between domino tilings and paths. Here we consider domino tiling of the rectangle R as a sequence of C_i-tilings. Every C_i-tiling is edge of graph $G(\mathbb{Z}_2^m, E_C)$. So number of all such sequences is the number of all possible sequences (paths) $v_0, v_1, \dots, v_n$, where $(v_{i-1}, v_i) \in E_{C_i}$ and $v_0 = v_n = \vec{0}$. Vertices v_0 and v_n should be $\vec{0}$, because left and right borders of the rectangle are straight and not intersected by any domino.

According to the property of adjacency matrices the number of such paths is equal to $(\vec{0}, \vec{0})$ entry of the matrix $A_{C_1} \cdot A_{C_2} \cdots A_{C_n}$. ■

For example, number of domino tilings of 3×3 square with removed $(2,2)$ square is equal to $(1,1)$ entry of the matrix $A_3 \cdot A_{1|\bar{1}|1} \cdot A_3$. Calculating:

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_{1|\bar{1}|1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So

$$A_3 \cdot A_{1|\bar{1}|1} \cdot A_3 = \begin{bmatrix} \boxed{2} & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Obviously, there are two possible domino tilings:

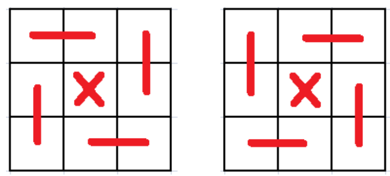


Figure 3: Domino tilings of 3×3 rectangle with $(2, 2)$ square removed

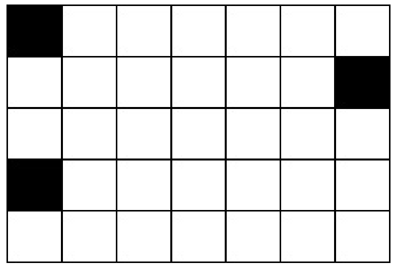
One more example, let R be 4×7 rectangle with four removed squares: $(1, 3), (4, 3), (1, 6), (2, 6)$. Then number of domino tilings of R is equal to $(1, 1)$ entry of this matrix:

$$A_4^2 \cdot A_{\bar{1}|2|\bar{1}} \cdot A_4^2 \cdot A_{\bar{2}|2} \cdot A_4 = \begin{bmatrix} \boxed{82} & 0 & 0 & 47 & 0 & 0 & 0 & 0 & 0 & 47 & 0 & 0 & 82 & 0 & 0 & 47 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 28 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 28 & 0 & 0 & 16 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 26 & 0 & 0 & 15 & 0 & 0 & 0 & 0 & 0 & 15 & 0 & 0 & 26 & 0 & 0 & 15 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 14 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 14 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 28 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 28 & 0 & 0 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 14 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 14 & 0 & 0 & 8 \end{bmatrix}$$

5 $(u, v)-$Rectangle

Definition 5.1. Let $u, v \in \mathbb{Z}_2^m$. Let m units wide rectangle with cross section u on the left board and with cross section v on the right board be called $(u, v)-$rectangle.

In other words, $(u, v)-$rectangle is rectangle with some removed squares on the first and the last columns. For example, this is $(10010, 01000)$-rectangle of length 7:



Definition 5.2. Let $t(u, v, n)$ be the number of $(u, v)-$rectangle of length n. And let

$$T(u, v, z) = \sum_{i=0} t(u, v, i) z^i$$

Remark. For $n = 0$ the sequence is defined in the following way:

$$t(u, v, 0) = \begin{cases} 1, & \text{if } u = v \\ 0, & \text{otherwise} \end{cases}$$

Theorem 5.1. *(u, v) entry of the matrix $(I - zA_m)^{-1}$ is equal to $T(u, v, z)$.*

Proof. It is clear that the following recurrence relation holds

$$t(u, v, n+1) = \sum_{(w,v) \in E_m} t(u, w, n)$$

Consequently,

$$T(u, v, z) = z \sum_{(w,v) \in E_m} T(u, w, z) + t(u, v, 0)$$

Let matrix T_m be a matrix $\{T(u, v, z)\}$. Then these recurrence relations can be represented in the matrix form:

$$T_m = zT_m A_m + I$$

Therefore,

$$T_m = (I - zA_m)^{-1}$$

■

6 Rectangle with one removed square

In the section 4 we have described method of calculating the number of domino tilings of $m \times n$ rectangle with some removed squares for certain natural values. Calculating numbers of domino tilings of $m \times n$ rectangle with exactly on removed square led us to the interesting fact. The number of domino tilings is the same if any of border squares is removed. For example,

4140081	0	4140081	0	4140081	0	4140081
0	2483688	0	2748768	0	2483688	0
4140081	0	4299761	0	4299761	0	4140081
0	2786136	0	3195736	0	2786136	0
4140081	0	4346673	0	4346673	0	4140081
0	2786136	0	3195736	0	2786136	0
4140081	0	4299761	0	4299761	0	4140081
0	2483688	0	2748768	0	2483688	0
4140081	0	4140081	0	4140081	0	4140081

Figure 4: Number written in every square of 9×7 rectangle indicates the number of domino tilings of the rectangle with corresponding square removed.

Theorem 6.1. *Number of domino tilings of* $(2m-1) \times (2n-1)$ *with one border square removed is equal to number of spanning trees of the graph* $P_m \times P_n$.

Proof. Let G denote the $(2m-1) \times (2n-1)$ lattice; its points are (i, j) for $0 \leq i \leq 2m-2, 0 \leq j \leq 2n-2)$. Call point (i, j) black if i and j are both even; red if both are odd; green if sum of them is odd. The black points form an $m \times n$ lattice graph . Let T be any spanning tree of H. Let $a = (0, 2s), 0 \leq s \leq n-1$ and $x \neq a$, a black point. Then there is a well-defined first edge on the path in connecting x to a and this contains a green point x'. Let y be a red point. In the lattice of red points, there is a unique path not crossing T which connects y to the outside boundary of the lattice G and on this, there is a first edge which contains a green point. Let y' be this green point.

The pairs $(x, x'), (y, y')$ form, as is easily verified, a 1-factor of $G - a$. Conversely, lot F be a 1-factor of $G - a$. Consider the set of those edges of which contain an edge of F. These form a spanning tree. In fact, the number of edges of F adjacent to black points is $mn - 1$, so contains $mn - 1 = |V(H)| - 1$ edges; it suffices to prove they do not form circuit. Suppose, by way of contradiction, that they form a circuit C. The number of points of G inside C is odd (this easily follows, e.g. by induction on the length of C) and so, F cannot match them, a contradiction.

Thus we have established a one-to-one correspondence between spanning trees of , which is $P_m \times P_n$ graph and 1-factors of G a i.e. domino tilings of $(2m-1)\times(2n-1)$ with one removed border square. ∎

6.1 Rectangle $3 \times (2n+1)$

Theorem 6.2. *Let $d(i, j)$ be the number of domino tilings of $3 \times (2n+1)$ rectangle with (i, j) square removed. Then*

$$d(1, 2j+1) = d(3, 2j+1) = a(n)$$

$$d(2, 2j) = 2a(n-j) \cdot a(j-1)$$

where

$$a(n) = \frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right)$$

And $d(i, j) = 0$ if $i + j$ is odd.

Proof. We consider this problem in this way. If (u, v) is a tiling of column with removed square, then domino tiling of $3 \times (2n+1)$ rectangle appears to be a join of domino tilings of $(\vec{0}, u)$-rectangle and $(v, \vec{0})$-rectangle. Thereby, number of domino tilings is equal to

$$t(\vec{0}, u, s) \cdot t(v, \vec{0}, 2n-s) \tag{1}$$

for any (u, v), where s is a length of $(\vec{0}, u)$-rectangle.

If square $(1, 2j+1)$ is removed from $3 \times (2n+1)$ rectangle, then $(2j+1)^{th}$ column has five possible $\overline{1}|2$-tilings:

$$(000, 000),\ (000, 011),\ (011, 000),\ (010, 001),\ (001, 010)$$

Two of them, $(010, 001)$ and $(001, 010)$ do not fit, because numbers of squares in both of (left and right) remaining parts of the rectangle would be odd. Hence domino tilings are impossible for these two cases. Shortly, $t(000, 010, 2j) = t(000, 001, 2j) = 0$.

Therefore, according to (1)

$$\begin{aligned} d(1, 2j+1) =& t(000, 000, 2j) \cdot t(000, 000, 2n-2j) + t(000, 011, 2j) \cdot \\ & \cdot t(000, 000, 2n-2j) + t(000, 000, 2j) \cdot t(011, 000, 2n-2j) \end{aligned} \tag{2}$$

Let $a(n)$ be z^{2n} coefficient of power series $\frac{1}{1-4z^2+z^4}$. Then

$$a(n) = \frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right) \tag{3}$$

According to the theorem 5.1:

$$T(000,000,z) = \frac{1-z^2}{1-4z^2+z^4},\ T(000,011,z) = \frac{z^2}{1-4z^2+z^4}$$

Consequently,

$$t(000,000,2j) = a(j) - a(j-1),\ t(000,011,2j) = t(011,000,2j) = a(j-1) \quad (4)$$

After substituting (4) to (2) we get

$$d(1,2j+1) = a(j)\cdot a(n-j) - a(j-1)\cdot a(n-j-1) \quad (5)$$

Then after using (3) to calculate (5) we have:

$$d(1,2j+1) = \frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right)$$

So, $d(1,2j+1) = d(3,2j+1) = a(n)$.

If square $(2,2j)$ is removed from $3\times(2n+1)$ rectangle, then $(2j)^{th}$ column has four possible $1|\overline{1}|1$-tilings:

$$(101,000),\ (000,101),\ (100,001),\ (001,100)$$

Because of the oddity of square numbers of remaining left and right part of the rectangle, cases $(101,000)$, $(000,101)$ are impossible. Therefore,

$$\begin{aligned} d(2,2j) = {} & t(000,001,2j-1)\cdot t(100,000,2n-2j+1)+ \\ & + t(000,100,2j-1)\cdot t(001,000,2n-2j+1) \end{aligned} \quad (6)$$

According to the theorem 5.1:

$$T(000,100,z) = T(000,001,z) = \frac{z}{1-4z^2+z^4}$$

Consequently,

$$\begin{aligned} & t(000,001,2j-1) = t(000,100,2j-1) = a(j-1), \\ & t(001,000,2n-2j+1) = t(100,000,2n-2j+1) = a(n-j) \end{aligned} \quad (7)$$

After substituting (7) to (6) we get:

$$d(2,2j) = 2a(n-j)\cdot a(j-1)$$

If we apply chess coloring to $3\times(2n+1)$ rectangle, number of black colored squares will be one more than number of white colored. Hence there is no perfect matching, if white colored square is removed, because every domino covers one white and one black squares exactly. Thus $d(i,j)=0$ if $i+j$ is odd. ■

Corollary 6.2.1. *The number of domino tilings of* $3\times(2n+1)$ *rectangle is the same for any square removed from the border.*

6.2 Rectangle $5 \times (2n+1)$

Theorem 6.3. *The number of domino tilings of $5 \times (2n+1)$ rectangle is the same for any square removed from the border and is equal to*

$$\frac{1}{\sqrt{105}\cdot 4^{n+1}}\Big[\left(\left(3+\sqrt{5}\right)\left(5+\sqrt{21}\right)\right)^{n+1}-\left(\left(3+\sqrt{5}\right)\left(5-\sqrt{21}\right)\right)^{n+1}- \\ -\left(\left(3-\sqrt{5}\right)\left(5+\sqrt{21}\right)\right)^{n+1}+\left(\left(\sqrt{5}-3\right)\left(\sqrt{21}-5\right)\right)^{n+1}\Big] \tag{8}$$

Proof. Let square $(1, 2j+1)$ be removed from $5 \times (2n+1)$ rectangle. Then $(2j+1)^{th}$ column has 13 possible $\overline{1}|4$-tilings:

$$(v_1,v_1),\ (v_1,v_4),\ (v_4,v_1),\ (v_1,v_{10}),\ (v_{10},v_1),\ (v_{13},v_1),\ (v_1,v_{13}), \\ (v_1,v_{16}),\ (v_{16},v_1),\ (v_7,v_{10}),\ (v_{10},v_7),\ (v_4,v_{13}),\ (v_{13},v_4) \tag{9}$$

where $v_i \in \mathbb{Z}_2^5$:

$$v_1=(00000),\ v_4=(00011),\ v_7=(00110), \\ v_{10}=(01001),\ v_{13}=(01100),\ v_{16}=(01111) \tag{10}$$

All other $\overline{1}|4$-tilings do not fit because of the parity of left and right parts of the rectangle. Therefore, according to (1)

$$d(1,2j+1)=\sum_{(u,v)\in S} t(\vec{0},u,2j)\cdot t(v,\vec{0},2n-2j) \tag{11}$$

where S is set of suitable $\overline{1}|4$-tilings listed in (9).

Let $b[n]$ be z^{2n} coefficient of power series $\frac{1}{1-15z^2+32z^4-15z^6+z^8}$. Then

$$b[n]=\frac{1}{\sqrt{105}\cdot 2^{2n+7}}\Big[\left(-16+3\sqrt{5}+5\sqrt{21}\right)\left(\left(3+\sqrt{5}\right)\left(5-\sqrt{21}\right)\right)^{n+1}+ \\ +\left(16-3\sqrt{5}+5\sqrt{21}\right)\left(\left(3+\sqrt{5}\right)\left(5+\sqrt{21}\right)\right)^{n+1}+ \\ +\left(16+3\sqrt{5}-5\sqrt{21}\right)\left(\left(\sqrt{5}-3\right)\left(\sqrt{21}-5\right)\right)^{n+1}+ \\ +\left(-16-3\sqrt{5}-5\sqrt{21}\right)\left(\left(3-\sqrt{5}\right)\left(5+\sqrt{21}\right)\right)^{n+1}\Big] \tag{12}$$

According to the theorem 5.1:

$$\begin{aligned}
T(\vec{0}, v_1, z) &= \frac{1 - 7z^2 + 7z^4 - z^6}{1 - 15z^2 + 32z^4 - 15z^6 + z^8} \\
T(\vec{0}, v_4, z) &= \frac{z^2(3 - 4z^2 + z^4)}{1 - 15z^2 + 32z^4 - 15z^6 + z^8} \\
T(\vec{0}, v_7, z) &= \frac{z^2(2 - 5z^2 + z^4)}{1 - 15z^2 + 32z^4 - 15z^6 + z^8} \\
T(\vec{0}, v_{10}, z) &= \frac{z^2 + z^4}{1 - 15z^2 + 32z^4 - 15z^6 + z^8} \\
T(\vec{0}, v_{13}, z) &= \frac{z^2(2 - 5z^2 + z^4)}{1 - 15z^2 + 32z^4 - 15z^6 + z^8} \\
T(\vec{0}, v_{16}, z) &= \frac{z^2 - z^6}{1 - 15z^2 + 32z^4 - 15z^6 + z^8}
\end{aligned} \tag{13}$$

Consequently,

$$\begin{aligned}
t(\vec{0}, v_1, 2n) &= b[n] - 7b[n-1] + 7b[n-2] - b[n-3] \\
t(\vec{0}, v_4, 2n) &= 3b[n-1] - 4b[n-2] + b[n-3] \\
t(\vec{0}, v_7, 2n) &= 2b[n-1] - 5b[n-2] + b[n-3] \\
t(\vec{0}, v_{10}, 2n) &= b[n-1] + b[n-2] \\
t(\vec{0}, v_{13}, 2n) &= 2b[n-1] - 5b[n-2] + b[n-3] \\
t(\vec{0}, v_{16}, 2n) &= b[n-1] - b[n-3]
\end{aligned} \tag{14}$$

After substituting (14) to (11) and calculating it with (12) we get:

$$\begin{aligned}
d(1, 2j+1) = &\frac{1}{\sqrt{105} \cdot 4^{n+1}} \Big[\left(\left(3 + \sqrt{5}\right)\left(5 + \sqrt{21}\right)\right)^{n+1} - \left(\left(3 + \sqrt{5}\right)\left(5 - \sqrt{21}\right)\right)^{n+1} - \\
&- \left(\left(3 - \sqrt{5}\right)\left(5 + \sqrt{21}\right)\right)^{n+1} + \left(\left(\sqrt{5} - 3\right)\left(\sqrt{21} - 5\right)\right)^{n+1} \Big]
\end{aligned} \tag{15}$$

Let square (3, 1) be removed from $5 \times (2n+1)$ rectangle. It is clear that

$$d(3, 1) = t(00100, 00000, 2n+1) \tag{16}$$

According to the theorem 5.1

$$T(00100, 00000, z) = \frac{z - z^5}{1 - 15z^2 + 32z^4 - 15z^6 + z^8} \tag{17}$$

Thus,

$$t(00100, 00000, 2n+1) = b[n] - b[n-2] \tag{18}$$

Drawing a conclusion from (16) and (19) and calculating using (12) lead us to:

$$\begin{aligned} d(3,1) = & \frac{1}{\sqrt{105 \cdot 4^{n+1}}} \Big[\left(\left(3+\sqrt{5}\right)\left(5+\sqrt{21}\right) \right)^{n+1} - \left(\left(3+\sqrt{5}\right)\left(5-\sqrt{21}\right) \right)^{n+1} - \\ & - \left(\left(3-\sqrt{5}\right)\left(5+\sqrt{21}\right) \right)^{n+1} + \left(\left(\sqrt{5}-3\right)\left(\sqrt{21}-5\right) \right)^{n+1} \Big] \end{aligned} \tag{19}$$

Since rectangle is self-similar figure $d(1, 2j+1) = d(5, 2j+1)$ and $d(3, 1) = d(3, 2n+1)$. Eventually,

$$d(1, 2j+1) = d(5, 2j+1) = d(3, 1) = d(3, 2n+1) = (8) \tag{20}$$

for any $j \in (1, n)$. ■

Theorem 6.4. *Let $d(i, j)$ be the number of domino tilings of $5 \times (2n+1)$ rectangle with (i, j) square removed. Then*

$$\begin{aligned} d(2,2j) = d(4,2j) = & \frac{1}{105 \cdot 2^{2n}} \Big[\left(1080 - 476\sqrt{5} + 236\sqrt{21} - 104\sqrt{105}\right) \left(\left(3-\sqrt{5}\right)\left(5+\sqrt{21}\right) \right)^{n-1} + \\ & + \left(1080 - 476\sqrt{5} - 236\sqrt{21} + 104\sqrt{105}\right) \left(\left(3-\sqrt{5}\right)\left(5-\sqrt{21}\right) \right)^{n-1} + \\ & + \left(1080 + 476\sqrt{5} - 236\sqrt{21} - 104\sqrt{105}\right) \left(\left(3+\sqrt{5}\right)\left(5-\sqrt{21}\right) \right)^{n-1} + \\ & + \left(1080 + 476\sqrt{5} + 236\sqrt{21} + 104\sqrt{105}\right) \left(\left(3+\sqrt{5}\right)\left(5+\sqrt{21}\right) \right)^{n-1} + \\ & + \sqrt{21} \left(\left(5-\sqrt{21}\right)^{n+1} - \left(5+\sqrt{21}\right)^{n+1} \right) \left(\left(3-\sqrt{5}\right)^{j-1} \left(3+\sqrt{5}\right)^{n-j} + \left(3+\sqrt{5}\right)^{j-1} \left(3-\sqrt{5}\right)^{n-j} \right) + \\ & + \sqrt{5} \left(\left(3-\sqrt{5}\right)^{n+1} - \left(3+\sqrt{5}\right)^{n+1} \right) \left(\left(5-\sqrt{21}\right)^{j-1} \left(5+\sqrt{21}\right)^{n-j} + \left(5+\sqrt{21}\right)^{j-1} \left(5-\sqrt{21}\right)^{n-j} \right) \Big] \end{aligned} \tag{21}$$

and

$$\begin{aligned} d(3,2j+1) = & \frac{1}{105 \cdot 4^{n+1}} \cdot \Big[4 \left(\left(3-\sqrt{5}\right)^{n+1} + \left(3+\sqrt{5}\right)^{n+1} \right) \left(\left(5-\sqrt{21}\right)^{j} \left(5+\sqrt{21}\right)^{n-j} + \left(5+\sqrt{21}\right)^{j} \left(5-\sqrt{21}\right)^{n-j} \right) - \\ & - 4 \left(\left(5-\sqrt{21}\right)^{n+1} + \left(5+\sqrt{21}\right)^{n+1} \right) \left(\left(3-\sqrt{5}\right)^{j} \left(3+\sqrt{5}\right)^{n-j} + \left(3+\sqrt{5}\right)^{j} \left(3-\sqrt{5}\right)^{n-j} \right) + \\ & + \left(11+\sqrt{105}\right) \left(\frac{1}{2} \left(\left(3-\sqrt{5}\right)\left(5-\sqrt{21}\right) \right)^{n+1} + \frac{1}{2} \left(\left(3+\sqrt{5}\right)\left(5+\sqrt{21}\right) \right)^{n+1} - \right. \\ & \left. - 2 \left(\left(3+\sqrt{5}\right)\left(5-\sqrt{21}\right) \right)^{j} \left(\left(3-\sqrt{5}\right)\left(5+\sqrt{21}\right) \right)^{n-j} - 2 \left(\left(3-\sqrt{5}\right)\left(5+\sqrt{21}\right) \right)^{j} \left(\left(3+\sqrt{5}\right)\left(5-\sqrt{21}\right) \right)^{n-j} \right) + \\ & + \left(11-\sqrt{105}\right) \left(\frac{1}{2} \left(\left(3+\sqrt{5}\right)\left(5-\sqrt{21}\right) \right)^{n+1} + \frac{1}{2} \left(\left(3-\sqrt{5}\right)\left(5+\sqrt{21}\right) \right)^{n+1} - \right. \\ & \left. - 2 \left(\left(3-\sqrt{5}\right)\left(5-\sqrt{21}\right) \right)^{j} \left(\left(3+\sqrt{5}\right)\left(5+\sqrt{21}\right) \right)^{n-j} - 2 \left(\left(3+\sqrt{5}\right)\left(5+\sqrt{21}\right) \right)^{j} \left(\left(3-\sqrt{5}\right)\left(5-\sqrt{21}\right) \right)^{n-j} \right) \Big] \end{aligned} \tag{22}$$

and $d(i, j) = 0$ if $i + j$ is odd.

Proof. Since square $(4, 2j)$ is symmetric to square $(2, 2j)$ in $5\times(2n+1)$ rectangle, obviously $d(2, 2j) = d(4, 2j)$. If square $(2, 2j)$ is removed from $5 \times (2n + 1)$, then according to chess coloring principle $(2j)^{th}$ column has ten possible $1|\overline{1}|3$-tilings:

$$(v_2, v_{17}),\ (v_{17}, v_2),\ (v_5, v_{17}),\ (v_{17}, v_5),\ (v_5, v_{20}),\\ (v_{20}, v_5),\ (v_2, v_{23}),\ (v_{23}, v_2),\ (v_8, v_{17}),\ (v_{17}, v_8) \tag{23}$$

where $v_i \in \mathbb{Z}_2^5$:

$$v_2 = (00001),\ v_5 = (00100),\ v_8 = (00111),\\ v_{17} = (10000),\ v_{20} = (10011),\ v_{23} = (10110) \tag{24}$$

Therefore, according to (1)

$$d(2, 2j) = \sum_{(u,v)\in S} t(\vec{0}, u, 2j - 1) \cdot t(v, \vec{0}, 2n - 2j + 1) \tag{25}$$

where S is set of suitable $1|\overline{1}|3$-tilings listed in (23). According to the theorem 5.1:

$$\begin{aligned} T(\vec{0}, v_2, z) =T(\vec{0}, v_5, z) = T(\vec{0}, v_{17}, z) &= \frac{z - z^5}{1 - 15z^2 + 32z^4 - 15z^6 + z^8} \\ T(\vec{0}, v_8, z) &= \frac{z - 4z^3 + 3z^5}{1 - 15z^2 + 32z^4 - 15z^6 + z^8} \\ T(\vec{0}, v_{20}, z) &= \frac{z - 5z^3 + 2z^5}{1 - 15z^2 + 32z^4 - 15z^6 + z^8} \\ T(\vec{0}, v_{23}, z) &= \frac{z^3 + z^5}{1 - 15z^2 + 32z^4 - 15z^6 + z^8} \end{aligned} \tag{26}$$

Consequently,

$$\begin{aligned} &t(\vec{0}, v_2, 2n + 1) = t(\vec{0}, v_5, n) = t(\vec{0}, v_{17}, n) = b[n] - b[n - 2] \\ &t(\vec{0}, v_8, 2n + 1) = b[n] - 4b[n - 1] + 3b[n - 2] \\ &t(\vec{0}, v_{20}, 2n + 1) = b[n] - 5b[n - 1] + 2b[n - 2] \\ &t(\vec{0}, v_{23}, 2n + 1) = b[n - 1] + b[n - 2] \end{aligned} \tag{27}$$

After substituting (27) to (25) and calculating it with (12) we get (21).

If square $(3, 2j + 1)$ is removed from $5 \times (2n + 1)$, then according to chess coloring principle $(2j + 1)^{th}$ column has 11 possible $2|\overline{1}|2$-tilings:

$$(v_1, v_1),\ (v_1, v_4),\ (v_4, v_1),\ (v_1, v_{25}),\ (v_{25}, v_1),\ (v_4, v_{25})\\ (v_{25}, v_4),\ (v_1, v_{28}),\ (v_{28}, v_1),\ (v_{10}, v_{19}),\ (v_{19}, v_{10}) \tag{28}$$

where $v_i \in \mathbb{Z}_2^5$:

$$\begin{aligned} &v_1 = (00000),\ v_4 = (00011),\ v_{10} = (01001),\\ &v_{19} = (10010),\ v_{25} = (11000),\ v_{28} = (11011) \end{aligned} \tag{29}$$

Therefore, according to (1)

$$d(2, 2j + 1) = \sum_{(u,v)\in S} t(\vec{0}, u, 2j) \cdot t(v, \vec{0}, 2n - 2j) \tag{30}$$

where S is set of suitable $2|\overline{1}|2$-tilings listed in (28). According to the theorem 5.1:

$$\begin{aligned} &T(\vec{0}, v_1, z) = \frac{1 - 7z^2 + 7z^4 - z^6}{1 - 15z^2 + 32z^4 - 15z^6 + z^8}\\ &T(\vec{0}, v_4, z) = T(\vec{0}, v_{25}, z) = \frac{z^2(3 - 4z^2 + z^4)}{1 - 15z^2 + 32z^4 - 15z^6 + z^8}\\ &T(\vec{0}, v_{10}, z) = T(\vec{0}, v_{19}, z) = \frac{z^2 + z^4}{1 - 15z^2 + 32z^4 - 15z^6 + z^8}\\ &T(\vec{0}, v_{28}, z) = \frac{z^2 - z^6}{1 - 15z^2 + 32z^4 - 15z^6 + z^8} \end{aligned} \tag{31}$$

Consequently,

$$\begin{aligned} &t(\vec{0}, v_1, 2n) = b[n] - 7b[n-1] + 7b[n-2] - b[n-3]\\ &t(\vec{0}, v_4, 2n) = t(\vec{0}, v_{25}, 2n) = 3b[n-1] - 4b[n-2] + b[n-3]\\ &t(\vec{0}, v_{10}, 2n) = t(\vec{0}, v_{19}, 2n) = b[n-1] + b[n-2]\\ &t(\vec{0}, v_{28}, 2n + 1) = b[n-1] - b[n-3] \end{aligned} \tag{32}$$

After substituting (32) to (30) and calculating it with (12) we get (22). ■

References

[1] Jordan Pollak,David Klarner, *Domino tiling with fixed width*, Discrete Math. **32**(1980), 45-52.

[2] A.M. Karavaev, S.N. Perepechko, *Generating functions in dimer problem on rectangular net graphs*, Information Processes, **13** (2013), No.4, p.374-400